Dynamical Constraints in Satellite Photogrammetry

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The feasibility of including rotational dynamic constraints in the process of satellite photogrammetric triangulation is demonstrated. An analytical solution and the corresponding analytic expressions for the partial derivatives with respect to initial conditions are given for the idealized case of the torque-free motion of a rigid triaxial satellite. The development of a state transition matrix solution for the direction cosines is included. This state transition matrix allows the satellite principal axes to be tracked relative to any arbitrary inertial reference frame. It also enables essential uncoupling of the partial derivatives with respect to initial Euler angles from those with respect to initial angular velocities. Comparative computer results for various sets of dynamic constraints are summarized.

Introduction

IN the process of satellite photogrammetric triangulation, the Earth-fixed object space coordinates of various Earth surface features are deduced from satellite photograph coordinates of the images of those features. The fundamental mathematical transformations central to this process are based solely on the principles of geometric optics and can be deduced from the grometry shown in Figs. 1 and 2.

These transformations are called the colinarity equations and can be written[1]

$$x_p = x_0 - f\left[\frac{C_{11}(X_p - X_c) + C_{12}(Y_p - Y_c) + C_{13}(Z_p - Z_c)}{C_{31}(X_p - X_c) + C_{32}(Y_p - Y_c) + C_{33}(Z_p - Z_c)}\right] \tag{1a}$$

$$y_p = y_0 - f\left[\frac{C_{21}(X_p - X_c) + C_{22}(Y_p - Y_c) + C_{23}(Z_p - Z_c)}{C_{31}(X_p - X_c) + C_{32}(Y_p - Y_c) + C_{33}(Z_p - Z_c)}\right] \tag{1b}$$

Equations (1) project the position of a point located at object space coordinates (X_p, Y_p, Z_p) into its image plane coordinates (x_p, y_p) for a camera with principal point offset (x_0, y_0) and focal length f. The perspective center of the camera is located at object space coordinates (X_c, Y_c, Z_c). The image space (x, y, z) axes are oriented relative to the object space (X, Y, Z) axes by the direction cosines $C_{\ell m}$; $\ell, m = 1,3$. Equations (1) can be written functionally as

$$x_p = F(X_p, Y_p, Z_p, X_c, Y_c, Z_c, \Phi, \Theta, \Psi, x_0, f) \tag{2a}$$

$$y_p = G(X_p, Y_p, Z_p, X_c, Y_c, Z_c, \Phi, \Theta, \Psi, y_0, f) \tag{2b}$$

where Φ, Θ, Ψ are the conventional 1-2-3 rotation sequence Euler angles parameterizing the direction cosines $C_{\ell m}$. If several points are considered in each of n photographs, Eqs. (1) and (2) can be doubly subscripted to denote the equations corresponding to the ith point appearing in the jth photograph as

$$x_{p_{ij}} = F(X_{p_i}, Y_{p_i}, Z_{p_i}, X_{c_j}, Y_{c_j}, Z_{c_j}, \Phi_j, \Theta_j, \Psi_j, x_{0_j}, f_j) \tag{3a}$$

$$y_{p_{ij}} = G(X_{p_i}, Y_{p_i}, Z_{p_i}, X_{c_j}, Y_{c_j}, Z_{c_j}, \Phi_j, \Theta_j, \Psi_j, y_{0_j}, f_j) \tag{3b}$$

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Equations (3) can be segregated into subsets depending on whether or not the particular point is a *control point* (i.e. one for which there exists *a priori* knowledge of its object space coordinates). Other points having distinctly measurable image plane coordinates in two or more photographs and whose object space coordinates are unknown and are called *pass points*. Depending upon the number of photographs in a strip, the number of strips with sidelap, the number and distribution of control points, the number of pass points and the manner in which they are shared by adjacent photographs, an intimidating variety of observations equations of the form (3) can be defined in practical applications. It is not uncommon to encounter several thousand such nonlinear equations containing several hundred unknowns. An obvious computational burden is associated with the least-squares solution of this class of problems.

In the traditional approach to this problem, the camera center coordinates $(X_{c_j}, Y_{c_j}, Z_{c_j})$ and camera orientation $(\Phi_j, \Theta_j, \Psi_j)$ for each photograph, as well as the pass-point coordinates have been treated as independent unknowns. The introduction of orbital dynamical constraints, conceived by Brown[2,3] and implemented by Light[4] and Hartwell,[5] allows the elimination of the $3n$ (n = number of photograhps) camera center coordinates in favor of six osculating orbital elements. The purpose of this paper is to carry the incorporation of dynamic constraints to its logical conclusion through rigorous satisfaction of the satellite equations of rotational motion.

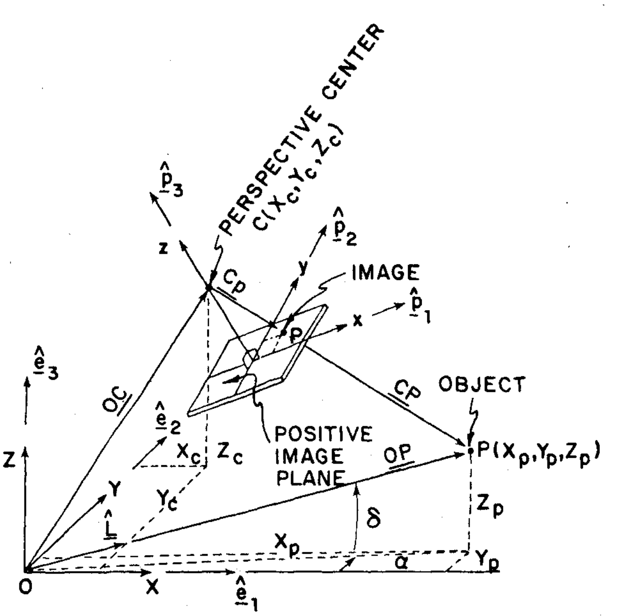

Fig. 1 Colinearity of perspective center, image, and object.

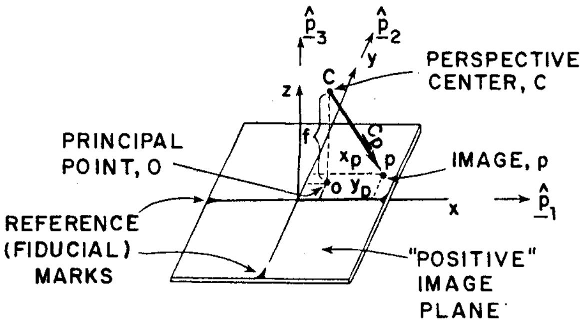

Fig. 2 Photographic coordinate system.

This process allows the further reduction of unknowns from $3n$ orientation angles to six osculating attitude elements or constants of the rotational motion. The specific mathematical model chosen here for the incorporation of attitude constraints is an analytic solution for the free motion of a general (triaxial) rigid body. This idealization may prove practical for some cases, but it is recognized that more complete dynamic models will be required to realize the ultimate practical potential of this approach. The osculating attitude elements chosen are the initial angular velocities and 1-2-3 Euler angles of the motion.

Orbital Dynamical Constraints

The essence of orbit constrained photogrammetry is the recognition that the camera exposure stations along a given strip of n photographs are dynamically constrained according to

$$X_{c_j} = X_c(t_j, c_1, c_2, \ldots, c_6) \tag{4a}$$

$$Y_{c_j} = Y_c(t_j, c_1, c_2, \ldots, c_6) \quad j = 1,2,\ldots,n, \tag{4b}$$

$$Z_{c_j} = Z_c(t_j, c_1, c_2, \ldots, c_6) \tag{4c}$$

where Eqs. (4) are functional representations of the solution of the spacecraft's translational equations of motion. The six constants ($c_1, \ldots, c_6$) can be any set of initial conditions or osculating orbital elements that uniquely define the orbit. The least-squares solution of the triangulation problem is modified to include the dynamic constraints as follows: current estimates of $c_1, c_2, \ldots, c_6$ are used in Eqs. (4) to compute X_{c_j}, Y_{c_j}, Z_{c_j} at each photographs exposure time t_j. These resulting current estimates of camera exposure coordinates are used in Eq. (3) together with the other parameter estimates

$$(X_{p_i}, Y_{p_i}, Z_{p_i}, \Phi_j, \Theta_j \Psi_j, x_{0_j}, y_{0_j}, f_j)$$

to determine current computed values of $x_{p_{ij}}$, $y_{p_{ij}}$ which are in turn differenced from the corresponding observed values to find the current residual vector. The partial derivatives of the observation equations (3) with respect to the orbital elements, $c_1, c_2, \ldots, c_6$ (the elements of the A matrix), are determined by the chain differentiation rule applied to Eq. (3) and (4). For example

$$\frac{\partial x_{p_{ij}}}{\partial c_1} = \frac{\partial F}{\partial X_{c_j}} \frac{\partial X_{c_j}}{\partial c_1} + \frac{\partial F}{\partial Y_{c_j}} \frac{\partial Y_{c_j}}{\partial c_1} + \frac{\partial F}{\partial Z_{c_j}} \frac{\partial Z_{c_j}}{\partial c_1}$$

The partial derivatives with respect to the other parameters are identical in form to the corresponding equations for unconstrained photogrammetric triangulation.

Depending on problem requirements, various specific forms of Eq. (4) are available and the corresponding partial derivatives readily obtainable. Further dicsussion of orbital constraints is found in the work of Brown,[2,3] Light,[4] and Hartwell.[5]

Attitude Dynamical Constraints

The inclusion of attitude constraints is completely analogous to orbital constraints. However, analytic solutions for rotational motion and the corresponding partial derivatives are considerably more complicated.

The free motion of a triaxial rigid body is governed by Euler's equations, which can be written

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 \tag{5a}$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_1 \omega_3 \tag{5b}$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 \tag{5c}$$

where I_1, I_2, I_3 are the moments of inertia relative to a mass-centered principal body axis system, denoted by the mutually orthogonal unit vectors ($\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$); ω_1, ω_2, ω_3 are $\{\hat{p}\}$ components of the inertial angular velocity of the body and the overdot denotes differentiation with respect to time.

Equations (5) can be solved analytically in terms of Jacobian elliptic functions (see Appendix A). This solution can be summarized functionally as

$$\omega_1(t) = \omega_1[t, t_0, I_1, I_2, I_3, \omega_1(t_0), \omega_2(t_0), \omega_3, (t_0)] \tag{6a}$$

$$\omega_2(t) = \omega_2[t, t_0, I_1, I_2, I_3, \omega_1(t_0), \omega_2(t_0), \omega_3(t_0)] \tag{6b}$$

$$\omega_3(t) = \omega_3[t, t_0, I_1, I_2, I_3, \omega_1(t_0), \omega_2(t_0), v3(t_0)] \tag{6c}$$

Specification of the attitude history of the motion requires another integration of kinematic expressions relating angular velocities to the time rate of change of Euler angles. This integration (for a general triaxial body) invariably involves an elliptic integral of the third kind. The usual integrations also require restrictions on the orientation of the inertial reference frame, i.e., one of the inertial axes must be aligned with the angular-momentum vector of the body. This special inertial reference frame is called the *angular momentum frame* and is denoted by unit vectors ($\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$). If the Euler angles (ϕ_1, ϕ_2, ϕ_3; called *solution angles*) that locate the principal body axes relative to the angular momentum frame represent a 1-2-3 rotation sequence, it is most convenient to choose the angular momentum frame so that $\hat{h}_1$ is aligned with the angular momentum vector. The analytical solution for this situation is given in Appendix B and can be summarized as

$$\phi_1(t) = \phi_1(t_0) + G_3 \Omega (t - t_0) - [\Phi_4(v, w_3) - \Phi_4(v_0, w_3)] \tag{7a}$$

$$\phi_2(t) = \sin^{-1}\left(\frac{I_2 \omega_3(t)}{H}\right), \qquad \frac{\pi}{2} \le \phi_2 \le \frac{\pi}{2} \tag{7b}$$

$$\phi_3(t) = \tan^{-1}\left(\frac{-I_2 \omega_2(t)}{I_1 \omega_1(t)}\right), \qquad \pi \le \phi_3 \le \pi \tag{7c}$$

Use of the angular-momentum reference frame greatly simplifies the analytic solution of the problem. Unfortunately, its introduction complicates the application of the solution in most practical situations where the motion relative to some more arbitrary inertial frame is required. If, for simplicity, the Earth is considered nonrotating in the current discussion, the direction cosines appearing in Eqs. (1) are required relative to the inertial reference frame (denoted by the unit vector triad $\hat{n}_1$, $\hat{n}_2$, $\hat{n}_3$), in which the object space coordinates are measured. It would appear to be necessary, therefore, to determine the relative orientation of $\{\hat{h}\}$ and $\{\hat{n}\}$. This problem can be eliminated and at the same time the partial derivatives of $\Phi_j(t), \Theta_j(t), \Psi_j(t)$ relative to the attitude initial conditions can be simplified through the use of a simple manipulative device. The three reference frames $\{\hat{p}\}$, $\{\hat{h}\}$, and $\{\hat{n}\}$ are related as follows

$$\{\hat{p}\} = [C(\omega,\phi,\kappa)]\{\hat{n}\} \tag{8a}$$

$$\{\hat{p}\} = [C(\phi_1,\phi_2,\phi_3)]\{\hat{h}\} \tag{8b}$$

$$\{\hat{h}\} = [C(\theta_1,\theta_2,\theta_3)]\{\hat{n}\} \tag{8c}$$

where ω,ϕ,κ constitute a set of 1-2-3 Euler angles relating $\{\hat{p}\}$ to $\{\hat{n}\}$; ϕ_1,ϕ_2,ϕ_3 defined as before, and $\theta_1,\theta_2,\theta_3$ are the three *constant* Euler angles relatively orienting the two inertial frames (θ_1, θ_2, θ_3 can represent any rotation sequence). Substitution of the right-hand side of Eq. (8c) for $\{\hat{h}\}$ in Eq. (8b) and comparison of the result with Eq. (8a) indicates that at any instant

$$[C(\omega(t),\ \phi(t),\ \kappa(t))] = [C(\phi_1(t),\ \phi_2(t),\ \phi_3(t))][C(\theta_1,\theta_2,\theta_3)] \tag{9}$$

This expression can be evaluated at the initial time t_0 and solved for $[C(\theta_1,\theta_2,\theta_3)]$ to find

$$[C(\theta_1,\theta_2,\theta_3)] = [C(\phi_1(t_0),\phi_2(t_0),\phi_3(t_0))]^T \times [C(\omega(t),\phi(t_0),\kappa(t_0))] \tag{10}$$

Equation (10) can be combined with Eq. (9) to give

$$[C(\omega(t),\phi(t),\kappa(t))] = [C(\phi_1(t),\phi_2(t),\phi_3(t))] \times [C(\phi_1(t_0),\phi_2(t_0),\phi_3(t_0))]^T \times [C(\omega(t_0),\phi(t_0),\kappa(t_0))] \tag{11}$$

Explicit dependence on θ_1, θ_2, $\theta_3$in Eq. (11) has been elminated in favor of the initial conditions on the solution angles ϕ_1, ϕ_2, ϕ_3 and on the initial (assumed known or measurable) relative orientation of $\{\hat{p}\}$ and $\{\hat{n}\}$. It is no longer necessary to explicitly specify the relative orientation of the two inertial frames [if desired, however, this information is directly obtainable from Eq. (10)]. A most important theoretical and practical significance of Eq. (11) is revealed by further consideration. Clearly at time $t=t_0$, the matrix product

$$[C(\phi_1(t),\phi_2(t),\phi_3(t))][C(\phi_1(t_0),\phi_2(t_0),\phi_3(t_0))]^T$$

becomes the identity matrix; furthermore, it can be shown by direct differentiation that this matrix product is independent of $\phi_1(t_0)$ at all times, and consequently has no possible dependence on $\omega(t_0)$, $\phi(t_0)$ or $\kappa(t_0)$ [see Eq. (7) and Appendix B]. It can therefore be concluded that Eq. (11) is in the form

$$[C(\omega(t),\ \phi(t),\kappa(t))] = [\Phi(t,t_0][C(\omega(t),\phi(t_0),\kappa(t_0))] \tag{12}$$

where $[\Phi(t,t_0)] = [C(\phi_1(t),\phi_2(t),\phi_3(t))]\ [C(\phi_1(t_0),\phi_2(t_0),\phi_3(t_0))]^T$ = the state transition matrix solution for the direction cosine matrix $[C(\omega,\phi,\kappa)]$. All of the important properties of a state transition matrix form of solution can now be used to advantage in this problem. Once $[\Phi(t,t_0)]$ is obtained from Eqs. (B5), (B10), and (B2), the direction cosine matrix relating the body axes orientation to an *arbitrary* inertial frame at time t can be determined from Eq. (12), given initial body orientation relative to that frame. Furthermore, any desired set of Euler angles at time t can be calculated by inserting the proper elements of the direction cosine matrix. Finally, since the state transition matrix is independent of $\phi_1(t_0)$, this initial condition can arbitrarily be chosen as zero.

This, in addition to the properties that

$$\frac{\partial[C(\omega(t),\phi(t),\kappa(t))]}{\partial\omega(t_0),\phi(t_0),\kappa(t_0)} = [\Phi(t,t_0)]\frac{\partial[C(\omega(t_0),\phi(t_0),\kappa(t_0))]}{\partial\omega(t_0),\phi(t_0),\kappa(t_0)} \tag{13}$$

and

$$\frac{\partial[C(\omega(t),\phi(t),\kappa(t))]}{\partial\omega_1(t_0),\ \omega_1(t_0),\omega_3(t_0)} = \frac{\partial[\Phi(t,t_0)]}{\partial\omega_1(t_0),\omega_2(t_0),\omega_3(t_0)}[C(\omega(t_0),\phi(t_0),\kappa(t_0))]$$

greatly reduce the effort required in finding the partial derivatives. See Appendix C for the partial derivatives of the rotational motion relative to attitude initial conditions.

In order to complete the incorporation of attitude constraints, it is necessary to account for the rotation of the Earth since the direction consines of Eq. (1) are measured relative to an Earth-fixed frame. If the arbitrary inertial frame $\{\hat{n}\}$ is assumed to be a geocentric nonrotating frame of a recent epoch, the transformation to the Earth fixed (object space) frame, denoted by $\{\hat{e}\}$, is a simple rotation about the Earth's polar axis, i.e.,

$$\{\hat{e}\} = [M(t)]\{\hat{n}\} \tag{14}$$

where

$$[M(t)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta = \dot{\theta}(t-t_0) + \theta_{\text{Greenwich at } t_0}$

$\dot{\theta}$ = constant angular rotational rate of the Earth

Combination of Eq. (14) and Eq. (8a) gives

$$\{\hat{p}\} = [C(\omega,\phi,\kappa)][M(t)]^T\{\hat{e}\} \tag{15}$$

The matrix product in Eq. (15) is identified as the required direction cosine matrix relating the imaging space to the object space, specifically,

$$[C(\Phi,\Theta,\Psi)] = [C(\omega,\phi,\kappa)][M(t)]^T \tag{16}$$

Since $[M(t)]$ does not depend on the initial conditions of the attitude motion the partial derivatives of $[C(\Phi,\Theta,\Psi)]$ with respect to initial conditions are given by

$$\frac{\partial[C(\Phi,\Theta,\Psi)]}{\partial p_i} = \frac{\partial[C(\omega,\phi,\kappa)]}{\partial p_i}[M(t)]^T,\quad i=1,\ldots,6 \tag{17}$$

where

$$p_1=\omega(t_0),\ p_2=\phi(t_0),\ p_3=\kappa(t_0),\ p_4=\omega_1(t_0),$$
$$p_5=\omega_2(t_0),\ p_6=\omega_3(t_0),$$

and $\partial[C(\omega,\phi,\kappa)]/\partial p_i$ are summarized in Eq. (13) and given explicitly in Appendix C.

Implementation of Dynamically Constrained Photogrammetry for a Simple Problem

In order to test the feaity of incorporating full dynamic constraintsts, an extremely idealized photogrammetry problem was created and analyzed. A rigid triaxial satellite ($I_1=3$, $I_2=2$, $I_3=1$) was placed in 100-mile circular polar

Table 1 A comparison of the effect of dynamical constraints on photogrammetric triangulation

Program	Max. pass-point coordinate error after convergence (true—converged) (ft)	Mean pass-point coordinate error (ft)	Standard deviation of errors (ft)	CDC 6400 central processor time (sec)
UCPHOTO	205.08	12.24	73.58	5.82
OCPHOTO	93.45	−6.36	37.47	5.66
DCPHOTO	−57.88	−1.60	25.54	6.86

orbit over a nonrotating Earth. Two-body translational motion and torque-free rotational motion were assumed. A single strip of four photographs, each with approximately 50% overlap was generated. These photographs covered a ground strip of approximately 100 miles by 250 miles.

Four control points and eight pass points were chosen on the ground strip and distributed in such a fashion as to give a total of 60 observation equations. The observation coordinates were truncated to simulate 1 mil measurement precision. It was further assumed that the camera focal length, f, the principal point offset (x_0, y_0) and the camera exposure times were perfectly known. Initial guesses of the pass-point coordinates and camera center coordinates were corrupted from their true values by an average of 2 miles. Initial guesses on camera axes Euler angles were corrupted by an average of 1.5°.

Three separate least-square differential correction programs were written. All computer runs were made on the tversity of Virginia CDC 6400 computer. These programs are summarized as follows.

UCPHOTO (Unconstrained Photogrammetric Triangulation)

All parameters appearing in the observation equations were treated as independent. The dimensions of the A matrix‡ were 60 (observations) × 48 (parameters), where $48 = 3$ times the number of pass points plus 3 times the number of camera center locations plus 3 camera axes Euler angles at each exposure time $= 3(8) + 3(4) + 3(4)$.

OCPHOTO (Orbit Constrained Triangulation)

Pass point coordinates, camera orientation at each time and orbit initial conditions were considered independent. The A matrix was dimensioned 60 (observations) × 42 (parameters), where $42 = 3$ times the number of pass points plus 6 orbit initial conditions plus 3 Euler angles for each photograph $3(8) + 6 + 3(4)$.

DCPOTO (Fully Dynamically Constrained Triangulation)

Pass point coordinates, orbit initial conditions and attitude initial conditions were considered independent. The dimensions of the A matrix were 60 (observations) × 36 (parameters) where $36 = 3$ times the number of pass points plus 6 orbit initial conditions plus 6 attitude initial conditions $= 3(8) + 6 + 6$.

The state transition matrix form of the attitude solution previously described, the angular velocity solution, the ϕ_1, ϕ_2, ϕ_3 solution, and the partial derivations given in the appendices were in DCPHOTO. The solution was verified by numerical integration and the partials were verified by finite differences. The essential results of these tests are summarized in Table 1. For this simple problem the best results were obtained using full dynamic consnts. Even through the dimension of the A matrix was reduced from 60×48 to 60×36, the computer run time was not decreased. This is because the matrix inversion savings were offset by the relatively elaborate dynamic model. The matrix calculation, however, dominate real world triangulation problems. Consequently, substantial cost reductions may be achieved in practical applications.

‡$A = \left[\dfrac{\partial(\text{measurements})}{\partial(\text{unknown parameters})} \right]$.

Concluding Remarks

The rotational dynamics model underlying the developments presented in this paper is based on the assumption that the motion is governed by the torque-free form of Euler's equations. This is in general a restrictive assumption, although there may be applications for which this model is adequate. Even in the presence of small torques, an osculating torque-free solution often provides good results over small elapsed times. In such situations, a net computational advantage might be obtained even if the solution has to be updated (i.e., new initial conditions found) every few photographs. For other problems, it may be possible to isolate non-torque-free/rigid-body effects in an Encke type perturbation treatment.[8] The solution and partial derivatives presented here could then be used as zeroth order approximations with the departure effects treated essentially separately. In-depth treatment of the Encke and other perturbation methods for more general dynamic models is left for subsequent investigations.

Due to the relative simplicity of the test problem described in the preceding section, little can be concluded from the results summarized in Table 1 except that the application of full dynamic constraints to photogrammetric triangulation appears feasible. The separate problem of bookkeeping associated with large real-life problems has not been addressed here but would essentially be unchanged by the inclusion of dynamic constraints. It is also impossible, at this point, to make any quantitative statements about a potential reduction of computer execution time for a large problem. The various tradeoffs, primarily between more computation required for the inclusion of dynamic constraints and the resulting less computation for the solution of the lower dimensioned least-squares normal equations, would have to be considered for each problem. The obvious advantages of the inclusion of dynamic constraints (given an adequate dynamic model of the motion) are the increased observability of the system and the numerical benefits ensuing from reduction in the number of unknowns in the problem. These advantages are evident even in the small problem discussed herein. In a very large problem, a solution might not otherwise be obtainable.

The description of the computer implementation details and documentation of numerical examples were rather constrained in the present paper. The interested reader is referred to Ref. 9; the computer software can be made available, upon request.

Appendix A: Summary of the Classical Solution for the Angular Velocities of a Torque-Free Rigid Body

Given: Principal moments of inertia I_1, I_2, I_3, initial time t_0, and initial angular velocities $\omega_1(t_0)$, $\omega_2(t_0)$, $\omega_3(t_0)$.

Find: $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$.

Assume: Principle body axes $\{p\}$ are ordered such that $I_1 \geq I_2 \geq I_3$,

Equations of Motion; Euler's Equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3 \tag{A1a}$$

$$\dot{\omega}_2 = \frac{-(I_1 - I_3)}{I_2}\omega_1\omega_3 \tag{A1b}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \tag{A1c}$$

Equations (A1) have two immediate first integrals corresponding to conservation of kinetic energy and angular momentum. There are, respectively

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \tag{A2}$$

and

$$H \cdot H = H^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \tag{A3}$$

where the angular momentum vector is

$$H = I_1\omega_1\hat{p}_1 + I_2\omega_2\hat{p}_2 + I_3\omega_3\hat{p}_3 \tag{A4}$$

It can be seen by taking the inertial derivative of H and using Euler's equations, that the direction of H as well as its magnitude is constant.

Equations (A2) and (A3) can be solved simultaneously for the squares of any two angular velocity components in terms of the square of the third and the constants I_1, I_2, I_3, $2T$, and H^2. The resulting expressions can be used with the squares of Eqs. (A1) to unc the equations of motion. Equations (A1) take the form

$$[\dot{\omega}_i]^2 = a_i + b_i\omega_i^2 + c_i\omega_i^4 \quad i = 1,2,3 \tag{A5}$$

$(a_i, b_i, c_i$ are constants)

Each of Eqs. (A5) can, via appropriate variable changes, be brought into one of the following forms

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = (1-x^2)(1-k^2x^2) \tag{A6a}$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = (1-x^2)(k'^2+k^2x^2) \tag{A6b}$$

or

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = (1-x^2)(x^2-k2) \tag{A6c}$$

These relationships are the differential equations that define the Jacobian elliptic functions, *sn, cn,* and *dn*, respectively.§ The parameter k is called the modulus and τ is proportional to time; k' is the complimentary modulus and is related to k by $k'^2 = 1-k^2$. The requirement implicit in Eq. (A6) that $0 \le k^2 \le 1$ gives rise to a branching in the angular velocity solution corresponding to whether or not $H^2 < 2I_2T$. When the considerable algebra necessary to bring Eq. (A5) into the form (A6) is completed, consistent with $0 \le k^2 \le 1$, the angular velocity solution can be summarized as follows

$$\omega_2(t) = \omega_{2m} sn[\Omega(t-t_2),\ k] \tag{A7}$$

$$\left.\begin{aligned} \omega_1(t) &= \omega_{1m} dn[\Omega(t-t_2), k] \\ \omega_3(t) &= \omega_{3m} cn[\Omega(t-t_2),\ k] \end{aligned}\right\} \text{ if } H^2 \ge 2I_2T \tag{A8a}$$

$$\left.\begin{aligned} \omega_1(t) &= \omega_{1m} cn[\Omega(t-t_2),\ k] \\ \omega_3(t) &= \omega_{3m} dn[\Omega(t-t_2)\ k] \end{aligned}\right\} \text{ if } H < 2I_2T \tag{A8b}$$

where ω_{im} are extreme values of $\omega_i (i = 1,2,3)$, $\omega_{im} = s_i|\omega_{im}|$, $s_i = +1, -1,$ t0, and

$$|\omega_{1m}| = \left[\frac{H^2 - 2I_3T}{I_1(I_1 - I_3)}\right]^{1/2} \tag{A9}$$

$$|\omega_{2m}| = \left[\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right]^{1/2} \tag{A10}$$

$$|\omega_{2m}| = \left[\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right]^{1/2} \text{if } H^2 \ge 2I_2T \tag{A11a}$$

$$= \left[\frac{H^2 - 2I_3T}{I_2(I_2 - I_3)}\right]^{1/2} \text{if } H^2 < 2I_2T \tag{A11b}$$

$$k = \left[\frac{(I_2 - I_3)(2I_1T - H^2)}{(I_1 - I_2)(H^2 - 2I_3T)}\right]^{1/2} \text{if } H^2 \ge 2I_2T \tag{A12a}$$

$$= \left[\frac{(I_1 - I_2)(H^2 - 2I_3T)}{(I_2 - I_3)(2I_1T - H^2)}\right]^{1/2} \text{if } H^2 < 2I_2T \tag{A12b}$$

$$\Omega = \left[\frac{(I_1 - I_2)(H^2 - 2I_3T)}{I_1I_2I_3}\right]^{1/2} \text{if } H^2 \ge 2I_2T \tag{A13a}$$

$$= \left[\frac{(I_2 - I_3)(2I_1T\text{`}TV)}{I_1I_2I_3}\right]^{1/2} \text{if } H^2 < 2I_2T \tag{A13b}$$

and t_2 = initial "phase" time.

It remains necessary to uniquely specify the sign s_i associated with the ω_{im} and to specify the phase time t_2. Since $\omega_2(t)$ is proportional to the *sn* function in both branches of the motion, Eq. (A7) can be evaluated at time t_0 and inverted to find

$$t_2 = t_0 - \frac{1}{\Omega} sn^{-1}\left[\frac{\omega_2(t_0)}{\omega_{2m}}\right] \tag{A14}$$

where

$$sn^{-1}\left[\frac{\omega_2(t_0)}{\omega_{2m}}\right] = F(\theta, k) \tag{A15}$$

$F(\theta,k)$ = incomplete elliptic integral of the first kind

$$\theta = \sin^{-1}\left[\frac{\omega_2(t_0)}{\omega_{2m}}\right] \tag{A16}$$

The use of the arcsin in Eq. (A16), however, introduces the possibility of quadrant difficulties in the determination of θ and consequently t_2. This problem is coupled with the choice of s_i since $\omega_{2m} = s_2|\omega_{2m}|$. If the decision process indicated in Table A1 is used to determine s_i, the phase angle θ is bounded by $\pm\pi/2$ and can conseqently be uniquely determined by Eq. (A16). This specification of s_i is also universal; that is, it gives the proper results in both branches and for all limiting cases of the motion (pure spin, body symmetry, and the branch transition).

Table A1 was deduced by assuming $-\pi/2 \le \theta \le \pi/2$ and imposing simultaneous consistency between Euler's equations

Table A1 Decision process

Initial conditions	Signs of ω_{im}
$\omega_1(t_0) \ne 0$ $\omega_2(t_0) \ne 0$ or $\omega_2(t_0) = 0$ $\omega_3(t_0) \ne 0$	$s_1 = \text{sign}(\omega_1(t_0))$ $s_3 = \text{sign}(\omega_3(t_0))$ $s_2 = -s_1s_3$
$\omega_1(t_0) = 0$ $\omega_2(t_0) \ne 0$ $\omega_3 \ne 0$	$s_3 = \text{sign}(\omega_3(t_0))$ $s_1 = -s_3\ \text{sign}(\omega_2(t_0))$ $s_2 = -s_1s_3$
$\omega_1(t_0) \ne 0$ $\omega_2(t_0) \ne 0$ $\omega_3(t_0) = 0$	$s_1 = \text{sign}(\omega_1(t_0))$ $s_3 = s_1\ (\omega_2(t_0))$ $s_2 = -s_1s_3$
Pure spin about either principal axis $\hat{p}_i$	$s_i = \text{sign}(\omega_i(t_0))$ $s_j = s_k = 0;\ i,j,k = 1,2,3$ $j,k \ne i$

§Compare with *Handbook of Elliptic Integrals for Engineers and Scientists*, P.F. Byrd and M.D. Friedman.

evaluated at t_0 and the derivatives with respect to τ of the elliptic functions evaluated at t_0, for all possible sign combinations of initial conditions.

Appendix B: Summary of the Solution for 1-2-3 Euler Angles Orienting Principle Body Axes to an Angular Momentum Reference Frame for the Free Motion of a General Rigid Body

Given: The principle inertias of the body I_1, I_2, I_3, initial time t_0, initial angular velocities $\omega_1(t_0)$, $\omega_2(t_0)$, $\omega_3(t_0)$, initial value of ϕ_1, the first Euler angle, and the angular velocity solution of Appendix A.

Find: $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, the 1-2-3 Euler angles as functions of time.

Consider the body mass centered principle axes $\{\hat{p}\}$ and the angular momentum inertial frames $\{\hat{h}\}$, chosen so that $\hat{h}_1 = (H/H)$ (Fig. 3)

The unit vector triads $\{\hat{p}\}$ and $\{\hat{h}\}$ are related by

$$\begin{Bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{Bmatrix} = [C] \begin{Bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{Bmatrix} \tag{B1}$$

where $[C]$ is the direction cosine matrix parameterized by ϕ_1, ϕ_2, ϕ_3 as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} C\phi_2 C\phi_3 & C\phi_1 S\phi_3 + S\phi_1 S\phi_2 C\phi_3 & S\phi_1 S\phi_3 - C\phi_1 S\phi_2 C\phi_3 \\ -C\phi_2 S\phi_3 & C\phi_1 C\phi_3 - S\phi_1 S\phi_2 S\phi_3 & S\phi_1 C\phi_3 + C\phi_1 S\phi_2 S\phi_3 \\ S\phi_2 & -S\phi_1 C\phi_2 & C\phi_1 C\phi_2 \end{bmatrix} \tag{B2}$$

where $S\phi_i = \sin(\phi_i)$, $C\phi_i = \cos(\phi_i)$, $i = 1,2,3$. The angular momentum vector with components taken in the two reference frames is

$$H = H\hat{h}_1 = I_1\omega_1\hat{p}_1 + I_2\omega_2\hat{p}_2 + I_3\omega_3\hat{p}_3 \tag{B3}$$

Combination of (B1) and (B3) gives the three component equations

$$I_1\omega_1 = H\cos\phi_2\cos\phi_3 \tag{B4a}$$

$$I_2\omega_2 = H(-\cos\phi_2\sin\phi_3) \tag{B4b}$$

$$I_3\omega_3 = H\sin\phi_2 \tag{B4c}$$

which can be inverted to find

$$\phi_2 = \sin^{-1}\frac{I_3\omega_3}{H}, \quad -\frac{\pi}{2} \le \phi_2 \le \frac{\pi}{2} \tag{B5a}$$

$$\phi_3 = \tan^{-1}\left(\frac{-I_2\omega_2}{I_1\omega_1}\right), \quad -\pi \le \phi_3 \le \pi \tag{B5b}$$

Thus, two of the Euler angles (ϕ_2 and ϕ_3) can be determined directly from the angular velocity solution. The differential equation for the remaining angle ϕ_1 can be found by observing from Eq. (B2) that

$$\phi_1 = \tan^{-1}(-c_{32}/c_{33}) \tag{B6}$$

Differentiation with respect to time gives

$$\dot{\phi}_1 = \frac{c_{32}\dot{c}_{33} - c_{33}\dot{c}_{32}}{c_{32}^2 + c_{33}^2} \tag{B7}$$

The kinematic relationship[6]

$$[\dot{C}] = [\tilde{\omega}]^T [C]$$

where

$$[\tilde{\omega}]^T = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_2 & 0 \end{bmatrix}$$

can be used with Eqs. (B7), (B5), and (B2) to give

$$\dot{\phi}_1 = H\frac{I_1\omega_1^2 + I_2\omega_2^2}{I_1^2\omega_1^2 + I_2^2\omega_2^2} \tag{B8}$$

Equation (B8) is purely kinematic in nature. Dynamics are introduced through the angular velocity solution of Appendix A. After making these substitutions, changing the independent variable from t to τ, and considerable algebra, Eq. (B8) can ultimately be brought to the form

$$\frac{d\phi_1}{d\tau} = \frac{H}{I_3\Omega} - \frac{H(I_1 - I_3)}{I_1 I_3\Omega}\,\frac{1}{1 + a_3^2 sn^2(\tau,k)} \tag{B9}$$

where

$$a_3^2 = \frac{I_3(2I_1T - H^2)}{I_1(H^2 - 2I_3T)}, \quad H^2 \ge 2I_2T$$

$$= \frac{I_3(I_1 - I_2)}{I_1(I_2 - I_3)}, \quad H^2 < 2I_2T$$

The integration of Eq. (B9) involves an elliptic integral of the third kind, arising from the second term on the right-hand side of the equation, and can be written in terms of theta functions as¶

$$\phi_1(t) = \phi_1(t_0) + G_3\Omega(t - t_0) - [\Phi_4(v, w_3) - \Phi_4(v_0, w_3)] \tag{B10}$$

where

$$G_3 = \frac{H}{I_1\Omega} - \frac{\pi}{2K} i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)}, \quad i = \sqrt{-1} \tag{B11}$$

K = the complete elliptic integral of the first kind

$$\equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{B12}$$

$$i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} = \frac{4\sum_{n=1}^{\infty}(-1)^n q^{n^2}\sinh 2nw_3}{1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cosh 2nw_3} \tag{B13}$$

¶This solution is taken from unpublished notes by Dr. H. S. Morton, Jr., Department of Engineering Science and Systems, University of Virginia. It has been verified by direct differentiation and by numerical integration.

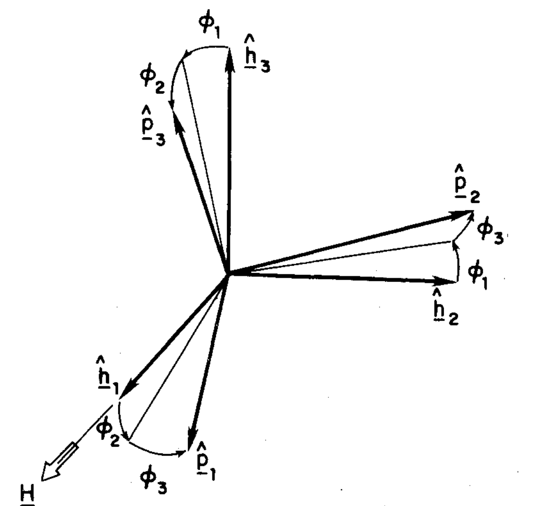

Fig. 3 1-2-3 Euler angle sequence.

$$q = \text{Jacobi's nome} \equiv e^{-\pi K'/k} \tag{B14}$$

$$K' \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k'^2 \sin^2\theta}} \tag{B15}$$

$$w_3 = \frac{\pi}{2K} F(\gamma, k') \tag{B16}$$

$F(\gamma,k')$ = incomplete elliptical integral of the first kind

$$= \int_0^{\gamma} \frac{d\theta}{\sqrt{1-k'^2 \sin^2\theta}} \tag{B17}$$

$$\gamma = \sin^{-1}\left[\frac{I_3(I_1-I_2)}{I_2(I_1-I_3)}\right]^{1/2}, \quad H^2 \geq 2I_2T \tag{B18}$$

$$= \sin^{-1}\left[\left[\frac{I_3(2I_1T-H^2)}{H(I_1-I_3)}\right]\right]^{1/2}, \quad H^2 < 2I_2T$$

$$\Phi_4(v, w_3) = \frac{1}{2i} \ln \frac{\theta_4(v+iw_3)}{\theta_4(v-iw_3)}$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^{2n}} \sin 2nv \sinh 2nw_3 \tag{B19}$$

$$v = (\pi/2K)\tau \tag{B20}$$

$$v_0 = (\pi/2K)\tau_0 \tag{B21}$$

and Ω, k, k', τ, τ_0 are defined as in Appendix *A*. It should be noted that the fundamental definition of the theta function, θ_4, is

$$\theta_4(u,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nu$$

While this solution appears quite intricate, it is very efficient once implemented. Rapidly converging arithmetic-geometric mean algorithms[7] can be used to compute K, K', and $F(\gamma,k')$. The infinite series of Eqs. (B13) and (B19) involve Jacobi's nome, q, which is typically small compared to unity (q approaches unity very slowly as the motion approaches the branch transition). Consequently, Eqs. (B13) and (B19) rarely require more than four or five terms for accurate convergence. Finally, many of the expressions [Eqs. (B11-B19), (B21) and $\Phi_4(v_0, w_0)$] are constants for a given solution and consequently need only be computed once.

Appendix C: The Partial Derivatives of the Rotational Motion of a General Rigid Body with Respect to Initial Conditions

Given: $I_1 \geq I_2 \geq I_3, \omega_1(t_0), \omega_2(t_0), \omega_3(t_0)$, $\omega(t_0), \phi(t_0)$, $\kappa(t_0)$, t_0, the angular velocity solution of Appendix A, the ϕ_1, ϕ_2, ϕ_3 solution (1-2-3) Euler angles relating the principal body frame $\{\hat{p}\}$ to the angular momentum frame $\{\hat{h}\}$ of Appendix B, and the state transition matrix solution (see text) for $[C(\omega(t), \phi(t), \kappa(t))]$.

Find:

$$\frac{\partial[C(\omega(t),\phi(t),\kappa(t))]}{\partial\omega(t_0),\phi(t_0),\kappa(t_0),\omega_1(t_0),\omega_2(t_0),\omega_3(t_0)}$$

Assume: All subsequent direction cosine matrices are rectified in terms of 1-2-3 Euler angles, and $\phi_1(t_0) = 0$ [see text below Eq. (1)].

Solution Summary:

Angular velocities;

$$\omega_1(t) = \omega_{1m} dn(\tau,k) \text{ if } H^2 \geq 2I_2T \tag{C1a}$$

$$\omega_1(t) = \omega_{1m} cn(\tau,k) \text{ if } H^2 > 2I_2T \tag{C1b}$$

$$\omega_2(t) = \omega_{2m} sn(\tau,k) \tag{C1c}$$

$$\omega_3(t) = \omega_{3m} cn(\tau,k) \text{ if } H^2 \geq 2I_2T \tag{C1d}$$

$$\omega_3(t) = \omega_{3m} dn(\tau,k) \text{ if } H^2 > 2I_2T \tag{C1e}$$

$$\tau = \Omega(t-t_2) \tag{C1f}$$

Solution angles;

$$\phi_1(t) = G_3\Omega(t-t_0) - [\Phi_4(v,w_3) - \Phi_4(v_0,w_3)] \tag{C2a}$$

$$\phi_2(t) = \sin^{-1}[I_3\omega_3(t)/H] \tag{C2b}$$

$$\phi_3(t) = \tan^{-1}[-I_2\omega_2(t)/I_1\omega_1(t)] \tag{C2c}$$

and

$$[C(\omega(t).\ \phi(t).\ \kappa(t))] = [\Phi(t,t_0)][C(\omega(t_0),\phi(t_0),\kappa(t_0))] \tag{C3}$$

where

$$[\Phi(t,t_0)] = [C(\phi_1(t),\phi_2(t),\phi_3(t))] \times [C(0,\phi_2(t_0),\ \phi_3(t_0))]^T \tag{C4}$$

For notational compactness the following abbreviations are made

$[C]$	$\equiv [C(\omega(t),\phi(t),\kappa(t))]$
$[C(t_0)]$	$\equiv [C(\omega(t_0),\phi(t_0),\kappa(t_0))]$
$[C(\phi)]$	$\equiv [C(\phi_1(t).\phi_2(t),\phi_3(t))]$
$[C(\phi(t_0))]^T$	$\equiv [C(0,\phi_2(t_0),\phi_3(t_0))]^T$
$sn\tau$	$\equiv sn(\tau,k)$
$cn\tau$	$\equiv cn(\tau,k)$
$dn\tau$	$\equiv dn(\tau,k)$
ω_i	$= \omega_1,\omega_2,\omega_3$

The partials of Eq. (C3) with respect to the initial Euler angles can be obtained immediately

$$\frac{\partial[C]}{\partial\omega(t_0),\phi(t_0),\kappa(t_0)} = [\phi(t,t_0)] \times \frac{\partial[C(t_0)]}{\partial\omega(t_0),\phi(t_0),\kappa(t_0)} \quad \text{(C5)}$$

where

$$\frac{\partial[C(t_0)]}{\partial\omega(t_0)} = \begin{bmatrix} 0 & -C_{13} & C_{12} \\ 0 & -C_{23} & C_{22} \\ 0 & -C_{33} & C_{32} \end{bmatrix}_{t=t_0} \quad \text{(C6a)}$$

$$\frac{\partial[C(t_0)]}{\partial\phi(t_0)} = \begin{bmatrix} -S\phi C\kappa & S\omega C\phi C\kappa & -C\omega C\phi C\kappa \\ S\phi S\kappa & -S\omega C\phi S\kappa & C\omega C\phi S\kappa \\ C\phi & S\omega S\phi & -C\omega S\phi \end{bmatrix}_{t=t_0} \quad \text{(C6b)}$$

(S and C denote sine and cosine, respectively)

$$\frac{\partial[C(t_0)]}{\partial\kappa(t_0)} = \begin{bmatrix} C_{21} & C_{22} & C_{23} \\ -C_{11} & -C_{12} & -C_{13} \\ 0 & 0 & 0 \end{bmatrix}_{t=t_0} \quad \text{(C6c)}$$

Since the dynamics enter through the angular velocity solution, the partials of Eq. (C3) with respect to the initial angular velocities are considerably more involved. Development of these derivatives can be outlined as follows. From Eq. (C3)

$$\frac{\partial[C]}{\partial\omega_i(t_0)} = \frac{\partial[\Phi(t,t_0)]}{\partial\omega_i(t_0)}[C(t_0) \quad \text{(C7)}$$

where from Eq. (C4),

$$\frac{\partial[\Phi(t,t_0)]}{\partial\omega_i(t_0)} = [C(\phi)]\frac{\partial[C(\phi(t_0))]^T}{\partial\omega_i(t_0)} + \frac{\partial[C(\phi)]}{\partial\omega_i(t_0)}[C(\phi(t_0))]^T \quad \text{(C8)}$$

The constant parts of Eq. (C8) can be determined using the solution angles [Eqs. (C2), and Appendix B] evaluated at time t_0;

$$\left.\begin{aligned} &\phi_1(t_0) = 0 \\ &\phi_2(t_0) = \sin^{-1}[I_3\omega_3(t_0)/H] \\ &\phi_3(t_0) = \tan^{-1}[-I_2\omega_2(t_0)/I_1\omega_1(t_0)] \end{aligned}\right\}$$

implies

$$\sin\phi_1(t_0) = 0, \quad \cos\phi_1(t_0) = 1$$
$$\sin\phi_2(t_0) = H_{b3}/H, \quad \cos\phi_2(t_0) = H_{12}/H$$
$$\sin\phi_3(t_0) = -H_{b2}/H_{12}, \quad \cos\phi_3(t_0) = H_{b1}/H_{12}$$

where

$$H_{bi} = I_i\omega_i(t_0) \qquad i=1,2,3$$
$$H_{12} = (H_{b1}^2 + H_{b2}^2)^{1/2}$$

It is now possible to write $[C(\phi(t_0))]^T$ in terms of the initial angular velocities and inertias as

$$[C(\phi(t_0))]^T = \begin{bmatrix} H_{b1}/H & H_{b2}/H & H_{b3}/H \\ -H_{b2}/H_{12} & H_{b1}/H_{12} & 0 \\ -H_{b1}H_{b3}/HH_{12} & -H_{b2}H_{b3}/HH_{12} & H_{12}/H \end{bmatrix} \quad \text{(C9)}$$

The derivatives of this matrix with respect to the initial angular velocities can be found directly. After considerable straightforward manipulation these partials can be written (letting $C_{\ell m}^T$ denote the elements of $[C(\phi(t_0))]^T$) as

$$\frac{\partial C_{11}^T}{\partial\omega_1} = \frac{I_1}{H^3}(H_{b2}^2 + H_{b3}^2) \quad \text{C10a)}$$

$$\frac{\partial C_{12}^T}{\partial\omega_1} = -\frac{I_1}{H^3}H_{b1}H_{b2} \quad \text{(C10b)}$$

$$\frac{\partial C_{13}^T}{\partial\omega_1} = -\frac{I_1}{H^3}H_{b1}H_{b3} \quad \text{(C10c)}$$

$$\frac{\partial C_{21}^T}{\partial\omega_1} = \frac{I_1}{H_{12}^3}H_{b1}H_{b2} \quad \text{(C10d)}$$

$$\frac{\partial C_{22}^T}{\partial\omega_1} = \frac{I_1}{H_{12}^3}H_{b1}^2 \quad \text{(C10e)}$$

$$\frac{\partial C_{23}^T}{\partial\omega_1} = 0 \quad \text{(C10f)}$$

$$\frac{\partial C_{31}^T}{\partial\omega_1} = \frac{I_1H_{b3}}{H^3H_{12}^3}[-H^2H_{12}^2 + H_{b1}^2(H^2 + H_{12}^2)] \quad \text{(C10g)}$$

$$\frac{\partial C_{32}^T}{\partial\omega_1} = \frac{I_1H_{b1}H_{b2}H_{b3}}{H^3H_{12}^3}(H^2 + H_{12}^2) \quad \text{(C10h)}$$

$$\frac{\partial C_{33}^T}{\partial\omega_1} = \frac{I_1H_{b1}H_{b3}^2}{H^3H_{12}} \quad \text{(C10i)}$$

$$\frac{\partial C_{11}^T}{\partial\omega_2} = -\frac{I_2}{H^3}H_{b1}H_{b2} \quad \text{(C11a)}$$

$$\frac{\partial C_{12}^T}{\partial\omega_2} = \frac{I_2}{H^3}(H_{b1} + H_{b3}) \quad \text{(C11b)}$$

$$\frac{\partial C_{13}^T}{\partial\omega_2} = -\frac{I_2}{H^3}H_{b2}H_{b3} \quad \text{(C11c)}$$

$$\frac{\partial C_{21}^T}{\partial\omega_2} = -\frac{I_2}{H_{12}^3}H_{b1}^2 \quad \text{(C11d)}$$

$$\frac{\partial C_{22}^T}{\partial\omega_2} = -\frac{I_2}{H_{12}^3}H_{b1}H_{b2} \quad \text{(C11e)}$$

$$\frac{\partial C_{23}^T}{\partial\omega_2} = 0 \quad \text{(C11f)}$$

$$\frac{\partial C_{31}^T}{\partial\omega_2} = \frac{I_2H_{b1}H_{b2}H_{b3}}{H^3H_{12}^3}(H^2 + H_{12}^2) \quad \text{(C11g)}$$

$$\frac{\partial C_{32}^T}{\partial\omega_2} = -\frac{I_2H_{b3}}{H^3H_{12}^3}[H^2H_{12}^2 - H_{b2}^2(H^2 + H_{12}^2) \quad \text{(C11h)}$$

$$\frac{\partial C_{33}^T}{\partial\omega_2} = \frac{I_2H_{b2}H_{b3}^2}{H^3H_{12}} \quad \text{(C11i)}$$

$$\frac{\partial C^T_{11}}{\partial \omega_3} = -\frac{I_3}{H^3} H_{b1} H_{b3} \quad \text{C12a)}$$

$$\frac{\partial C^T_{12}}{\partial \omega_3} = -\frac{I_1}{H^3} H_{b2} H_{b3} \quad \text{(C12b)}$$

$$\frac{\partial C^T_{13}}{\partial \omega_3} = \frac{I_3}{H^3} H^2_{12} \quad \text{(C12c)}$$

$$\frac{\partial C^T_{21}}{\partial \omega_3} = \frac{\partial C^T_{22}}{\partial \omega_3} = \frac{\partial C^T_{23}}{\partial \omega_3} = 0 \quad \text{(C12d)}$$

$$\frac{\partial C^T_{31}}{\partial \omega_3} = -\frac{I_3}{H^3} H_{b1} H_{12} \quad \text{(C12e)}$$

$$\frac{\partial C^T_{32}}{\partial \omega_3} = -\frac{I_3}{H^3} H_{b2} H_{12} \quad \text{(C12f)}$$

$$\frac{\partial C^T_{33}}{\partial \omega_3} = -\frac{I_3}{H^3} H_{b3} H_{12} \quad \text{(C12g)}$$

The remaining derivatives in Eq. (C8), i.e., the partials of the direction cosine matrix of solution angles at time t with respect to initial angular velocities, can be found from the chain rule,

$$\frac{\partial[C(\phi)]}{\partial \omega_i(t_0)} = \frac{\partial[C(\phi)]}{\partial \phi_1(t)} \frac{\partial \phi_1(t)}{\partial \omega_i(t_0)} + \frac{\partial[C(\phi)]}{\partial \phi_2(t)} \frac{\partial \phi_2(t)}{\partial \omega_i(t_0} + \frac{\partial[C(\phi)]}{\partial \phi_3(t)} \frac{\partial \phi_3(t)}{\partial \omega_i(t_0)} \quad \text{(C13)}$$

where the partial derivatives of $[C(\phi)]$ with respect to $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ are identical in form to Eqs. (C6) with $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$ substituted for $\omega(t_0)$, $\phi(t_0)$, and $\kappa(t_0)$, respectively. In order to evaluate Eq. (C13) it remains necessary to find the derivatives of the solution angles with respect to the initial rates. Since $\phi_2(t)$ and $\phi_3(t)$ are determined solely by the angular velocitity solution, the partials of these angles will be considered first.

$$\phi_2(t) = \sin^{-1}\left[\frac{I_3\omega_3(t)}{H}\right]$$

$$\frac{\partial \phi_2(t)}{\partial \omega_i(t_0)} = \frac{I_3}{H^2 \cos\phi_2(t)} \left[H \frac{\partial \omega_3(t)}{\partial \omega_i(t_0)} - \omega_3(t) \frac{\partial H}{\partial \omega_i(t_0)}\right] \quad \text{(C14)}$$

where

$$\frac{\partial H}{\partial \omega_i(t_0)} = \frac{I_i^2 \omega_i(t_0)}{H} \qquad i = 1,2,3 \quad \text{(C15)}$$

$$\phi_3(t) = \tan^{-1}\left[\frac{-I_2\omega_2(t)}{I_1\omega_1(t)}\right] \quad \text{(C16a)}$$

$$\frac{\partial \phi_3(t)}{\partial \omega_i(t_0)} = \frac{I_1 I_2}{I_1^2 \omega_1^2(t) + I_2^2 \omega_2^2(t))} \times \left[\omega_2(t) \frac{\partial \omega_1(t)}{\partial \omega_i(t_0)} - \omega_1(t) \frac{\partial \omega_2(t)}{\partial \omega_i(t_0)}\right] \quad \text{(C16b)}$$

Before considering the derivatives of $\phi_1(t)$ it is convenient at this point to turn to the angular velocity solution of Appendix A and develop the partial derivatives of the angular velocities at time t with respect to the initial angular velocities. Differentiation of Eqs. (C1) gives

$$\begin{aligned} \frac{\partial \omega_1(t)}{\partial \omega_i(t_0)} &= \frac{\partial \omega_{1m}}{\partial \omega_i(t_0)} dn\tau + \omega_{1m} \frac{\partial dn\tau}{\partial \omega_i(t_0)} \quad \text{if } H^2 \geq 2I_2T \\ &= \frac{\partial \omega_{1m}}{\partial \omega_i(t_0)} cn\tau + \omega_{1m} \frac{\partial cn\tau}{\partial \omega_i(t_0)} \quad \text{if } H^2 < 2I_2T \end{aligned} \quad \text{(C17)}$$

$$\frac{\partial \omega_2(t)}{\partial \omega_i(t_0)} = \frac{\partial \omega_{2m}}{\partial \omega_i(t_0)} sn\tau + \omega_{2m} \frac{\partial sn\tau}{\partial \omega_i(t_0)} \quad \text{(C18)}$$

$$\begin{aligned} \frac{\partial \omega_3(t)}{\partial \omega_i(t_0)} &= \frac{\partial \omega_{3m}}{\partial \omega_i(t_0)} cn\tau + \omega_{3m} \frac{\partial cn\tau}{\partial \omega_i(t_0)} \quad \text{if } H^2 \geq 2I_2T \\ &= \frac{\partial \omega_{3m}}{\partial \omega_i(t_0)} dn\tau + \omega_{3m} \frac{\partial dn\tau}{\partial \omega_i(t_0)} \quad \text{if } H^2 < 2I_2T \end{aligned} \quad \text{(C19)}$$

where

$$\frac{\partial \omega_{1m}}{\partial \omega_i(t_0)} = s_1 \frac{H \dfrac{\partial H}{\partial \omega_i(t_0)} - I_3 \dfrac{\partial T}{\partial \omega_i(t_0)}}{[I_1(I_1 - I_3)(H^2 - 2I_3T)]^{1/2}} \quad \text{(C20)}$$

$$\begin{aligned} \frac{\partial \omega_{2m}}{\partial \omega_i(t_0)} &= s_2 \frac{I_1 \dfrac{\partial T}{\partial \omega_i(t_0)} - H \dfrac{\partial H}{\partial \omega_i(t_0)}}{[I_2(I_1 - I_2)(2I_1T - H^2)]^{1/2}} \quad \text{if } H^2 < 2I_2T \\ &= s_2 \frac{H \dfrac{\partial H}{\partial \omega_i(T_0)} - I_3 \dfrac{\partial T}{\partial \omega_i(t_0)}}{[I_2(I_2 - I_3)(H^2 - 2I_3T)]^{1/2}} \end{aligned} \quad \text{(C21)}$$

$$\frac{\partial \omega_{3m}}{\partial \omega_i(t_0)} = s_3 \frac{I_1 \dfrac{\partial T}{\partial \omega_i(t_0)} - H \dfrac{\partial H}{\partial \omega_i(t_0)}}{[I_3(I_1 - I_3)(2I_1T - H^2)]^{1/2}} \quad \text{(C22)}$$

where

$$\frac{\partial T}{\partial \omega_i(t_0} = I_i \omega_i(t_0) \quad \text{(C23)}$$

$$\frac{\partial sn\tau}{\partial \omega_i(t_0)} = \frac{\partial sn\tau}{\partial \tau} \frac{\partial \tau}{\partial \omega_i(t_0)} + \frac{\partial sn\tau}{\partial k} \frac{\partial k}{\partial \omega_i(t_0)} \quad \text{(C24)}$$

$$\frac{\partial cn\tau}{\partial \omega_\theta(t_0)} = \frac{\partial cn\tau}{\partial \tau} \frac{\partial \tau}{\partial \omega_i(t_0)} + \frac{\partial cn\tau}{\partial k} \frac{\partial k}{\partial \omega_i(t_0)} \quad \text{(C25)}$$

$$\frac{\partial dn\tau}{\partial \omega_i(t_0)} = \frac{\partial dn\tau}{\partial \tau} \frac{\partial \tau}{\partial \omega_i(t_0)} + \frac{\partial dn\tau}{\partial k} \frac{\partial k}{\partial \omega_i(t_0)} \quad \text{(C26)}$$

$$\frac{\partial \tau}{\partial \tau_i(t_0)} = (t - t_0) \frac{\partial \Omega}{\partial \omega_i(t_0)} + \frac{\partial \tau_0}{\partial \omega_i(t_0)} \quad \text{(C27)}$$

where $\tau_0 = \Omega(t_0 - t_2)$.

If $H^2 \geq 2I_2T$,

$$\frac{\partial k}{\partial \omega_i(t_0)} = \left[\frac{(I_2 - I_3)}{(I_1 - I_3)(2I_1T - H^2)(H^2 - 2I_3T)^3}\right]^{1/2} \times \left[(H^2 - 2I_3T)\left(I_1 \frac{\partial T}{\partial \omega_i(t_0)} - H \frac{\partial H}{\partial \omega_i(t_0)}\right) - (2I_1T - H^2)\left(H \frac{\partial H}{\partial \omega_i(t_0)} - I_3 \frac{\partial T}{\partial \omega_i(t_0)}\right)\right] \quad \text{(C28a)}$$

If $H^2 < 2I_2T$

$$\frac{\partial k}{\partial \omega_i(t_0)} = \left[\frac{(I_1 - I_2)}{(I_2 - I_3)(H^3 - 2I_3T)(2I_1T - H^2)^3} \right]^{1/2}$$

$$\times \left[(2I_1T - H^2)(H\frac{\partial H}{\partial \omega_i(t_0)} - I_3\frac{\partial T}{\partial \omega_i(t_0)}) - (H^2 - 2I_3T)(I_1\frac{\partial T}{\partial \omega_i(t_0)} - H\frac{\partial H}{\partial \omega_i(t_0)}) \right] \tag{C28b}$$

If $H^2 \geq 2I_2T$,

$$\frac{\partial \Omega}{\partial \omega_i(t_0)} = \left[H\frac{\partial H}{\partial \omega_i(t_0)} - I_3\frac{\partial T}{\partial \omega_i(t_0)} \right] \left[\frac{(I_1 - I_2)}{I_1I_2I_3(H^2 - 2I_3T)} \right]^{1/2} \tag{C29a}$$

If $H^2 < 2I_2T$,

$$\frac{\partial \Omega}{\partial \omega_i(t_0)} = \left[I_1\frac{\partial T}{\partial \omega_i(t_0)} - H\frac{\partial H}{\partial \omega_i(t_0)} \right] \left[\frac{(I_2 - I_3)}{I_1I_2I_3(2I_1T - H^2)} \right]^{1/2} \tag{C29b}$$

$$\frac{\partial \tau_0}{\partial \omega_i(t_0)} = \frac{\partial F(\theta,k)}{\partial \theta}\frac{\partial \theta}{\partial \omega_i(t_0)} + \frac{\partial F(\theta,k)}{\partial k}\frac{\partial k}{\partial \omega_i(t_0)} \tag{C30}$$

where $\theta = \sin^{-1}[\omega_2(t_0)/\omega_{2m}]$

$$\frac{\partial \theta}{\partial \omega_i(t_0)} = \frac{1}{\cos\theta}\left[\frac{1}{\omega_{2m}}\frac{\partial \omega_2(t_0)}{\partial \omega_i(t_0)} - \frac{\omega_2(t_0)}{\omega_{2m}^2}\frac{\partial \omega_{2m}}{\partial \omega_i(t_0)} \right] \tag{C31}$$

$$\frac{\partial F(\theta,k)}{\partial \theta} = \frac{1}{(1 - k^2\sin^2\theta)^{1/2}} \tag{C32}$$

$$\frac{\partial F(\theta,k)}{\partial k} = \frac{E(\theta,k) - k'^2F(\theta,k)}{kk'^2} = \frac{k\sin\theta\cos\theta}{k'^2(1 - k^2\sin^2\theta)^{1/2}} \tag{C33}$$

where $E(\theta,k) \equiv$ incomplete elliptic integral of the second kind $\equiv$

$$E(\theta,k) \equiv \int_0^\theta \sqrt{1 - k^2\sin^2}\ d,$$

and
$$\frac{\partial sn\tau}{\partial \tau} = cn\tau dn\tau \tag{C34}$$

$$\frac{\partial cn\tau}{\partial \tau} = -sn\tau dn\tau \tag{C35}$$

$$\frac{\partial dn\tau}{\partial \tau} = -k^2 sn\tau cn\tau \tag{C36}$$

$$\frac{\partial sn\tau}{\partial k} = \frac{dn\tau cn\tau}{kk'^2}[-E(\tau) + k'^2\tau + k^2 sn\tau cn\tau/dn\tau] \tag{C37}$$

$$\frac{\partial cn\tau}{\partial k} = \frac{sn\tau dn\tau}{kk'^2}[E(\tau) - k'^2\tau - k^2 sn\tau cn\tau/dn\tau] \tag{C38}$$

$$\frac{\partial dn\tau}{\partial k} = \frac{k sn\tau cn\tau}{k'^2}[E(\tau) - k'^2\tau - dn\tau sn\tau/cn\tau] \tag{C39}$$

where $E(\tau) \equiv$ incomplete elliptic integral of the second kind.

$$E(\tau) \equiv \int_0^\tau dn^2\tau d\tau$$

Equations (C17) through (C39) provide the partial derivatives of the angular velocity solution which are required in Eqs. (C14) and (C16). Attention can now be returned to the first solution angle, ϕ_1. Differentiation of the first of Eqs. (C2) gives

$$\frac{\partial \phi_1(t)}{\partial \omega_i(t_0)} = \frac{\partial G_3}{\partial \omega_i(t_0)}\Omega(t - t_0) + G_3\frac{\partial \Omega}{\partial \omega_i(t_0)} - \frac{\partial \Phi_4(v,w_3)}{\partial \omega_i(t_0)} + \frac{\partial \Phi_4(v_0,w_3)}{\partial \omega_i(t_0)} \tag{C40}$$

(Refer to Appendix B for definitions of symbols.)

$$\frac{\partial G_3}{\partial \omega_i(t_0)} = \frac{1}{I_1\Omega^2}\left[\Omega\frac{\partial H}{\partial \omega_i(t_0)} - H\frac{\partial \Omega}{\partial \omega_i(t_0)} \right]$$

$$\frac{\pi}{2K^2}\, i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)}\frac{\partial K}{\partial \omega_i(t_0)} - \frac{\pi}{2K}\frac{\partial}{\partial \omega_i(t_0)}\left[i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} \right] \tag{C41}$$

$\partial H/\partial \omega_i(t_0)$ and $\partial \Omega/\partial \omega_i(t_0)$ are given by Eqs. (C15) and (C29), respectively.

$$\frac{\partial K}{\partial \omega_i(t_0)} = \frac{\partial K}{\partial k}\frac{\partial k}{\partial \omega_i(t_0)} = \frac{E - k'^2K}{kk'^2}\frac{\partial k}{\partial \omega_i(t_0)} \tag{C42}$$

$$\frac{\partial}{\partial \omega_i(t_0)}\left[i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} \right] = \frac{\partial}{\partial w_3}\left[i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} \right]\frac{\partial w_3}{\partial \omega_i(t_0)} + \frac{\partial}{\partial q}\left[i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} \right]\frac{\partial q}{\partial \omega_i(t_0)} \tag{C43}$$

The partials of $[i(\theta_4'(iw_3)/\theta_4(iw_3))]$ with respect to w_3 and q can be determined from term-by-term differentiation of the infinite series (B13):

$$\frac{\partial}{\partial w_3}\left[i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} \right] = \frac{S_1S_2 - S_3^3}{S_1^2} \tag{C44}$$

where

$$S_1 = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cosh 2nw_3 \tag{C45}$$

$$S_2 = 8\sum_{n=1}^{\infty}(-1)^n n^2 q^{n^2}\cosh 2nw_3 \tag{C46}$$

$$S_3 = 4\sum_{n=1}^{\infty}(-1)^n n q^{n^2}\sinh 2nw_3 \tag{C47}$$

and

$$\frac{\partial}{\partial q}\left[i\frac{\theta_4'(iw_3)}{\theta_4(iw_3)} \right] = \frac{S_1S_4 - S_3S_5}{S_1^2} \tag{C48}$$

where

$$S_4 = 4\sum_{n=1}^{\infty}(-1)^n n^3 q^{n^2-1}\sinh 2nw_3 \tag{C49}$$

$$S_5 = 2\sum_{n=1}^{\infty}(-1)^n n^2 q^{n^2-1}\cosh 2nw_3 \tag{C50}$$

$$\frac{\partial w_3}{\partial \omega_i(t_0)} = \frac{\pi}{2K}\left\{\frac{\partial F(\gamma,k')}{\partial \gamma}\frac{\partial \gamma}{\partial \omega_i(t_0)} + \left[\frac{\partial F(\gamma,k')}{\partial k} - \frac{F(\gamma,k')}{K}\frac{\partial K}{\partial k}\right]\frac{\partial k}{\partial \omega_i(t_0)}\right\} \quad \text{(C51)}$$

$$\frac{\partial F(\gamma,k')}{\partial \gamma} = \frac{1}{\sqrt{1-k'^2\sin^2\gamma}} \quad \text{(C52)}$$

$$\frac{\partial F(\gamma,k')}{\partial k} = \frac{\partial F(\gamma,k')}{\partial k'}\frac{\partial k'}{\partial k} \quad \text{(C53)}$$

$$\frac{\partial F(\gamma,k')}{\partial k'} = \frac{E(\gamma,k') - k^2F(\gamma,k')}{k'k^2} - \frac{k'\sin\gamma\cos\gamma}{k^2\sqrt{1-k'^2\sin^2\gamma}} \quad \text{(C54)}$$

$$\frac{\partial k'}{\partial k} = -\frac{k}{k'} \quad \text{(C55)}$$

$$\frac{\partial \gamma}{\partial \omega_i(t_0)} = 0 \text{ if } H^2 \geq 2I_2T = \left[\frac{I_1I_3}{(2I_1T-H^2)(H^2-2I_3T)}\right]^{1/2} \left[\frac{\partial T}{\partial \omega_i(t_0)} - \frac{2T}{H}\frac{\partial H}{\partial \omega_i(t_0)}\right], \quad \text{(C56)}$$

if $H^2 < 2I_2T$.

$$\frac{\partial q}{\partial \omega_i(t_0)} = \frac{\partial q}{\partial k}\frac{\partial k}{\partial \omega_i(t_0)} = \frac{\pi^2 q}{2kk'^2K^2}\frac{\partial k}{\partial \omega_i(t_0)} \quad \text{(C57)}$$

$$\frac{\partial \Phi_4(v,w_3)}{\partial \omega_i(t_0)} = \frac{\partial \Phi_4(v,w_3)}{\partial v}\frac{\partial v}{\partial \omega_i(t_0)} + \frac{\partial \Phi_4(v,w_3)}{\partial w_3}\frac{\partial w_3}{\partial \omega_i(t_0)} + \frac{\partial \Phi_4(v,w_3)}{\partial q}\frac{\partial q}{\partial \omega_i(t_0)} \quad \text{(C58)}$$

$\partial w_3/\partial \omega_i(t_0)$ and $\partial q/\partial \omega_i(t_0)$ given by Eqs. (C51) and (C57), respectively.

$$\frac{\partial v}{\partial \omega_i(t_0)} = \frac{\pi}{2K^2}\left[K\frac{\partial \tau}{\partial \omega_i(t_0)} - \tau\frac{\partial K}{\partial \omega_i(t_0)}\right] \quad \text{(C59)}$$

$\partial \tau/\partial \omega_i(t_0)$ and $\partial K/\partial \omega_i(t_0)$ given by Eqs. (C27) and (C42), respectively.

The partials of Φ_4 with respect to v, w_3, and q are found by differentiating the infinite series Eq. (B19)

$$\frac{\partial \Phi_4(v,w_3)}{\partial v} = 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\cos 2nv \sinh 2nw_3 \quad \text{(C60)}$$

$$\frac{\partial \Phi_4(v,w_3)}{\partial w_3} = 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin 2nv \cosh 2nw_3 \quad \text{(C61)}$$

and

$$\frac{\partial \Phi_4(v,w_3)}{\partial q} = 2\sum_{n=1}^{\infty}\frac{q^{n-1}(1+q^{2n})}{(1-q^{2n})^2}\sin 2nv \sinh 2nw_3 \quad \text{(C62)}$$

$\partial \Phi_4(v_0,w_3)/\partial \omega_i(t_0)$ is determined by expressions identical in form to Eqs. (C58)-(C62) with v_0 and τ_0 substituted for v and τ. This completes the equations necessary to calculate the partial derivatives of the direction cosine matrix with respect to initial conditions of the rotational motion.

This is indeed an intimidating collection of equations. However, as was the case with the solution itself, many of these partial derivatives are constant and need to be computed only once for a given motion solution. Most of the more intricate expressions fall into this category, e.g.

$$\frac{\partial[C(\phi(t_0)]^T}{\partial \omega_i(t_0)}, \quad \frac{\partial G_3}{\partial \omega_i(t_0)}, \quad \frac{\partial \Phi_4(v_0,w_3)}{\partial \omega_i(t_0)}$$

Many other terms are obtainable directly from the motion solution. The elliptic integrals can be efficiently computed using arithmetic-geometric mean algorithms and the series, $S_1, S_2, \ldots, S_5$, generally converge rapidly due to the presence of Jacobi's nome, q. Finally, this form of solution and partial derivatives can be easily modified to allow any rotational sequence of Euler angles to be chosen as initial conditions [Eq. C6)] are changed to the appropriate form and all other equations remain unchanged). Reference 9 provides complete documentation of the computer implementation of the above formulation.

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References

[1]Thompson, M.M., *Manual of Photogrammetry,* American Society of Photogrammetry, Vol. 1, 1966, p. 469.

[2]Brown, D., "A Unified Lunar Control Network," *Photogrammetric Engineering,* Vol. 34, 1968, pp. 1272-1292.

[3]Brown, D., "The Rigorous and Simultaneous Adjustment of Lunar Orbital Photography, Rome Air Development Center, Rome, N.Y., Report RADC-TR-70, 274, 1970.

[4]Light, D.L., "Photo Geodesy from Apollo," *Photogrammetric Engineering,* Vol. 38, 1972, pp. 547-587.

[5]Hartwell, J.G., et al., "A Photogrammetric and Tracking Network Analysis Program," (FOTONAP), U.W. Army Engineers Topographic Laboratories Report ETL-CR-73-17, 1973.

[6]Likins, P., *Elements of Engineering Mechanics,* McGraw-Hill, N.Y., 1973, p. 105.

[7]Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions,* National Bureau of Standards AMS-55, 9th Ed., pp. 569-607.

[8]Kraige, L.G. and Junkins, J.L., "Perturbation Formulations for Satellite Attitude Dynamics," *Celestial Mechanics,* Vol. 13, 1976, pp. 39-64.

[9]Blanton, J.N., "A New Formulation of the Free Motion of Triaxial Rigid Bodies and Applications to Satellite Photogrammetric Triangulation," Ph.D. Dissertation, University of Virginia, Charlottesville, Virginia, Aug. 1976.